INTRODUCTION TO ANALYTIC AND FORMAL GEOMETRY

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ABSTRACT. These are a preliminary draft of my notes for the lectures in Grenoble. They may be incomplete or they may contain some mistakes. Please feel free to comment or help to improve them.

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1. INTRODUCTION

These are notes of my five hours lectures held at the Summer school Foliations and Algebraic Geometry held at the Institut Fourier in Grenoble (F) in June 2019.

In the first part of the notes we explain some cornerstone theorems of algebraic geometry over the complex numbers. The proofs - and the statements - of these theorems use in a essential way methods coming from complex analysis and differential topology.

These theorems are:

– The Siegel Theorem on the field of meromorphic functions of a compact complex variety. This theorem tells us that the field of meromorphic functions of a compact complex variety has transcendence degree at most the dimension of the variety. Observe that the compactness

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property is necessary: even the field of meromorphic function of a one dimensional disk or an
affine variety has infinite transcendence degree over the complex number.

The proof uses some complex analysis (the maximum modulus property of analytic func-
tions) and the interaction between meromorphic functions and line bundles.

The result is very classical and its proof may be found for instance in [1].

– The Chow Theorem on analytic sub varieties of the projective spaces.

In this Theorem, one shows that a smooth analytic subvariety of a projective space is
actually algebraic (it is the zero set of a system of polynomial equations). This Theorem has
many proofs, the one proposed here is a simple adaptation of the proof of the Siegel Theorem.
It actually uses the same tools of it.

Both the theorems above use only tools from complex analysis and algebraic geometry. If
we allow to use also tools from differential topology - actually basic Morse theory - we can
give a reasonably self contained proof of:

– The Lefschetz Theorem on the fundamental group of hyperplane sections.

This Theorem tell us that, if $D$ is a smooth hyperplane section of a smooth projective
variety of dimension at least three, then the natural map between the fundamental group of
$D$ and the fundamental group of $X$ is an isomorphism. The proof we present here - due to
Andreotti and Frankel - is a classical application of the standard Morse theory thus it uses
tools from differential geometry and from algebraic topology.

One can read a proof of this theorem for instance in [6]

When one look to technics and the tools used to prove these theorems, even if some of them
may be stated in a quite general contest, one realizes that they cannot be easily translated in
more general contests. For instance, one can state the Lefschetz Theorem over arbitrary field
or even over arbitrary schemes but Morse Theory is no more available.

Formal Geometry is a tool which allows to use technics which look like analysis and topol-
yogy in the contest of general algebraic geometry. It essentially uses the fact that an analytic
function is known when we know its Taylor expansion. Following this principle, formal geom-
etry may be seen as an avatar of the germ of a analytic neighbourhood of a subscheme of
a scheme.

After a brief introduction to the general machinery of formal geometry -one can find a
more detailed introduction in [5] or [4] - we will explain a theorem, due to Grothendieck,
which plays the rôle of the Chow Theorem in this contest. We will end the lectures with
the description of a formal scheme which is not ”algebrizable” (which means that it not the
formal neighbourhood of a closed subscheme of a scheme).

In these lectures we do not mention possible applications of formal geometry. We would
like to quote some of them:

– One of the main application is in deformation theory: Usually one want to deform a
geometric structure (a scheme, a sheaf, etc) defined over a field, over a more general base (for
instance over a discrete valuation ring). In order to do this, one first deform it over artinian
rings and builds up a formal deformation of the object. Grothendieck algebraization theorem
allows, under some conditions, to prove that the formal deformation is actually ”algebraic”
by this we mean that there is a scheme the completion of which is the formal scheme we constructed.

An important application of formal geometry is an algebraic proof of an analogue of the Lefschetz hyperplane sections theorem for the fundamental group and for the Picard groups. One can give a proof of these theorems in the general theory of schemes (which for instance holds in positive characteristic). A posteriori one can see some similarities with the Andreotti–Frankel proof of it.

Another application of formal geometry is the "algebraic study" of differential equations. A differential equation on a general scheme is usually called a "foliation". Over analytic varieties, regular differential equations always have local solution. In the Zariski topology this is not anymore true. Nevertheless one can see that, at least in characteristic zero, there is always a solution in the formal geometry. In positive characteristic this is still true under some hypothesis (called $p$-closure condition).

These lectures are introductory to the lectures by J.-B. Bost. His lectures will cover more advanced topics in formal geometry. In particular he will develop some of the themes quoted above.

2. The theorem of Siegel on the field of meromorphic functions

2.1. The field of meromorphic functions of a variety. Let $U$ be an open set of $\mathbb{C}^n$. Denote by $\Gamma(U, \mathcal{O})$ the ring of holomorphic functions over it.

A meromorphic function $F$ on $U$ is a function $f$ on the complement of a nowhere dense subset $S$ of $U$ with the property that there exists a covering $U = \bigcup U_i$ and holomorphic functions $f_i$ and $g_i$ in $\Gamma(U_i, \mathcal{O})$ such that $g_i \cdot F|_{U_i \setminus S} = f_i|_{U_i \setminus S}$.

The set of meromorphic functions on $U$ is a field which will be denoted by $\mathcal{M}_U$.

By construction, given a meromorphic function $F$ on $U$ and a point $z \in U$, we can find an open disk $\Delta \subseteq U$ centred in $z$ and two relatively prime holomorphic functions $f$ and $g$ on $\Delta$ with the property that $F|_{\Delta} = \frac{f}{g}$. The functions $f$ and $g$ are well defined up to units. The analytic subset $Z(f)$ and $Z(g)$ depend only on $F$ and not on the choice of $f$ and $g$. Consequently, given a meromorphic function on $U$, we may define the zero locus of $F$ as the analytic subset given locally by $Z(f)$ and the pole locus of $F$ as the analytic subset locally given by $Z(g)$.

Remark 2.1. On a U.F.D. $A$, two elements $a$ and $b$ are said to be relatively prime if the class of $a$ is not a zero divisor in $A/(b)$ (the definition do not depend on the order or $a$ and $b$).

Observe that a point $z \in U$ may be at the same time a zero and a pole of a meromorphic function: for instance, $(0,0)$ is a zero and a pole of the function $F = \frac{x}{y}$ on $\mathbb{C}^2$.

Let $X$ be a smooth analytic variety defined over the field of the complex numbers.

The sheaf $M_X$ of meromorphic functions of $X$ is the sheaf which associates to every open set $U$ of $X$ the field $\mathcal{M}_U$. 
We define $\mathcal{M}_X$ to be the field $H^0(X, M_X)$.

2.2. Meromorphic functions and line bundles. We prove now a theorem which shows how a meromorphic function may be interpreted as a holomorphic function on a line bundle.

As in the case of a open set of $\mathbb{C}^n$, given a meromorphic function $F$ on an analytic variety $X$ we can define the set $Z(F)$ of zeroes of $F$ and the set $P(F)$ of poles of it.

We define $\text{Ind}(F)$ to be the analytic subset $Z(F) \cap P(F)$.

**Example 2.2.** Suppose that $X = \mathbb{P}^n$ with homogeneous coordinates $x_0 \ldots x_n$. If $P(x_0, \ldots, x_n)$ and $Q(x_0, \ldots, x_n)$ are two homogeneous non zero polynomials (with no common factors) of the same degree $d$, then the function $F := \frac{P}{Q}$ defines a meromorphic function on $X$. In particular we obtain that $\mathcal{M}_{\mathbb{P}^n} = \mathbb{C}(X_1, \ldots, X_n)$.

One can easily prove (exercise) that

**Proposition 2.3.** Let $f : X \to Y$ be an analytic dominant map between smooth analytic varieties and $F \in \mathcal{M}_Y$. Then $f^*(F)$ is a meromorphic function on $X$. In particular we have an inclusion $f^* : \mathcal{M}_Y \hookrightarrow \mathcal{M}_X$.

In particular we obtain:

**Proposition 2.4.** Suppose that $Z \subset X$ is an analytic subset of $X$ and $U_Z := X \setminus Z$. Let $\iota_{U_Z} : U_Z \hookrightarrow X$ the inclusion. Then the natural inclusion $\iota^* : \mathcal{M}_{U_Z} \hookrightarrow \mathcal{M}_X$ is an isomorphism.

As a corollary we obtain the following, very important:

**Proposition 2.5.** Let $f : X \to Y$ be an analytic proper map between smooth analytic varieties with the property that there exists proper analytic subvarieties $Z \subset X$ and $W \subset Y$ such that $f|_{U_Z} : U_Z \to U_W$ is an isomorphism. Then the induced map $f^a|_{U_Z} : \mathcal{M}_Y \to \mathcal{M}_X$ is an isomorphism.

Maps like the ones in the Proposition above are called modifications. Proposition tells us that $\mathcal{M}_X$ is invariant by modifications.

and

**Proposition 2.6.** Suppose that $f : X \to \mathbb{P}^1$ is an analytic dominant map and let $f^* : \mathcal{C}(X) \hookrightarrow \mathcal{M}_X$ the induced inclusion. Then $f$ determines a meromorphic function $f^*(X)$ which is well defined up to a scalar.

Let’s see how the last proposition may be almost inverted:

**Proposition 2.7.** Let $F \in \mathcal{M}_X$ be a non constant meromorphic function. Then we can find a closed analytic set $Z$ of codimension at least two and a dominant analytic map $h : U_Z \to \mathbb{P}^1$, such that $h^*(X) = F$.

Observe that we used Proposition 2.6 to identify $\mathcal{M}_{U_Z}$ and $\mathcal{M}_X$. 
Proof. Let \( Z := \text{Ind}(F) \). We can cover \( U_Z \) by open sets \( U_i \) in such a way that we can find coprime holomorphic functions \( f_i \) and \( g_i \) on \( U_i \) such that \( F|_{U_i} = \frac{f_i}{g_i} \). The analytic functions \( h_i : U_i \to \mathbb{P}^1 \) defined by \( h(z) := [f_i(z) : g_i(z)] \) glue together to an analytic function \( h : U_Z \to \mathbb{P}^1 \). Since, by construction \( h_i^*(X) = F|_{U_i} \), the conclusion follows. \( \square \)

A similar proof gives

**Theorem 2.8.** Let \( F_1, \ldots, F_n \) be \( \mathbb{C} \) linearly independent meromorphic functions on \( X \), then we can find an analytic subset \( Z \subset X \) of codimension at least two (possibly empty) and an analytic map \( h : U_Z \to \mathbb{P}^n \) such that \( h^*(X_i) = F_i \).

Let’s recall the statement of Hartogs Theorem

**Theorem 2.9.** (Hartogs) Let \( U \) be a open set in \( \mathbb{C}^n \) and \( Z \subset U \) be a closed analytic set of codimension at least two. Then every analytic function \( f \) defined over \( U \setminus Z \) extends uniquely to an analytic function on \( U \).

Hartogs Theorem can be stated in the following way: Under the hypothesis of Theorem 2.9 the natural restriction map

\[
\Gamma(U, \mathcal{O}) \longrightarrow \Gamma(U \setminus Z, \mathcal{O})
\]

is an isomorphism.

As a direct corollary of Hartogs Theorem we get

**Corollary 2.10.** Let \( \mathcal{L} \) be a line bundle on \( X \) and let \( Z \subset X \) be an analytic subset of codimension at least two. Then the natural restriction map

\[
H^0(X, \mathcal{L}) \longrightarrow H^0(U_Z, \mathcal{L})
\]

is an isomorphism.

Another corollary is the following:

**Corollary 2.11.** Suppose that \( Z \subset X \) is an analytic subset of codimension at least two, then the natural restriction map

\[
\iota : \text{Pic}(X) \longrightarrow \text{Pic}(U_Z)
\]

is an isomorphism.

Proof. The zero locus of an analytic function is an analytic set of codimension one. Consequently, suppose that \( U \) is an open set of \( X \) and \( g \) is a non vanishing analytic function on \( U \setminus Z \), then \( g \) extends (by Hartogs) to an analytic function on \( U \) which is non vanishing too. Indeed otherwise its zero locus would be contained in \( Z \) which is at least of codimension two and this is not possible.

The map \( \iota \) is injective: Suppose that \( \iota(\mathcal{L}) \) is the trivial line bundle, this implies that \( \iota(L) \) has a nowhere vanishing global section. This global section, by Hartogs, extends to global section of \( L \) which is nowhere vanishing. Thus \( \mathcal{L} \) is isomorphic to the trivial line bundle.
The map $\iota$ is surjective: Let $\mathcal{M}$ be a line bundle on $U_Z$. We may suppose that it is defined by a covering $\{U_i\}$ of $X$ and a cocycle $\{g_{ij}\}$ with $g_{ij}$ nowhere vanishing functions on $U_i \cap U_j \setminus Z$ (and satisfy the usual cocycle condition). Each $g_{ij}$ extends to a nowhere vanishing function on $U_i \cap U_j$ and the cocycle condition is preserved because it holds on dense subsets. Thus $\mathcal{M}$ extends to a line bundle on $X$. □

Suppose that $F_0, \ldots, F_n$ are meromorphic functions over a complex variety $X$, by Theorem 2.8 we get an analytic map $h : U_Z \to \mathbb{P}^n$ where $Z$ is an analytic subset of codimension at least two. Thus we obtain a line bundle $\mathcal{L}_{U_Z} := h^*(\mathcal{O}(1))$. The natural inclusion $H^0(\mathbb{P}^n, c\mathcal{O}(1)) \subset H^0(U_Z, \mathcal{L}_{U_Z})$ give rise to a $n + 1$ dimensional subspace $V$ of $H^0(U_Z, \mathcal{L}_{U_Z})$ and a surjective map

$$V \otimes \mathcal{O}_{U_Z} \to \mathcal{L}_{U_Z} \tag{2.4}$$

The line bundle $\mathcal{L}_{U_Z}$ can be extended to a line bundle $\mathcal{L}$ on $X$ and the subspace $V$ extends to a subspace of global sections of it.

Observe that the restriction map from $V$ to $\mathcal{L}$ is not surjective anymore.

To resume we obtain

**Theorem 2.12.** We can associate to $n$ $\mathbb{C}$–linearly independent meromorphic functions $F_1, \ldots, F_n$ on $X$ a line bundle $\mathcal{L}$ and a $n + 1$ dimensional subspace $V$ of $H^0(X, \mathcal{L})$. Viceversa, we can associate to a given a line bundle $\mathcal{L}$ and a subspace $V$ of dimension $n + 1$ of $H^0(X, \mathcal{L})$ a closed set $Z$ of codimension at least two and an analytic map $h : U_Z \to \mathbb{P}^n$ such that $h^*(\mathcal{O}(1)) = \mathcal{L}_{U_Z}$. Thus if $h^* : \mathbb{C}(X_1, \ldots, X_n) \to \mathcal{M}_X$ is the induced map, $h^*(X_i) = F_i$ are $n$ $\mathbb{C}$–linearly independent meromorphic functions on $X$. These constructions are one the inverse of the other.

**Remark 2.13.** If $F_1, \ldots, F_n$ are meromorphic functions on $X$ and $V$ is the space of holomorphic sections of the line bundle $\mathcal{L}$ associated to them, then we can find a basis $s_0, \ldots, s_n$ of $V$ such that, over the open set $s_0 \neq 0$, we have $F_i = \frac{s_i}{s_0}$.

### 2.3. Siegel Theorem.

In this paragraph we would like to prove the following

**Theorem 2.14.** Let $X$ be a smooth compact analytic variety of dimension $n$ then

$$\text{Trdeg}_\mathbb{C}(\mathcal{M}_X) \leq n. \tag{2.5}$$

Theorem 2.14 will be consequence of the following Theorem which has its own interest

**Theorem 2.15.** Let $X$ be a smooth analytic variety of dimension $n$ and $L$ be a line bundle over it. Then we can find a positive constant $C$ such that, for every positive integer $d$ we have

$$h^0(X, L^\otimes d) \leq C \cdot d^n. \tag{2.6}$$

As it is customary, we denote by $h^0(X, L)$ the dimension of the space of global sections $\Gamma(X, L)$. 

Let’s show first how Theorem 2.15 implies Theorem 2.14: It suffices to prove that, if $F_1,\ldots,F_{n+1}$ are non zero meromorphic sections over $X$, then we can find a polynomial $P(X_1,\ldots,X_{n+1}) \in \mathbb{C}[X_1,\ldots,X_{N+1}]$ such that $P(F_1,\ldots,F_{n+1}) = 0$ identically on $X$.

By Remark 2.13, we can find line bundle $L$ on $X$, a vector space $V \subset H^0(X,L)$ of dimension $N + 2$, a basis $s_0,\ldots,s_{N+1}$ and a dense open set $U \subseteq X$ where $s_0 \neq 0$, each of the $F_i$ has no poles and $F_i|_U = \frac{s_i}{s_0}$. Moreover, each of the $F_i$ is holomorphic on $U$.

For every $I := (i_0,\ldots,i_{n+1}) \in \mathbb{N}^{n+2}$ such that $i_0 + \cdots + i_{n+1} = d$ the product $s^I := s_0^{i_0} \cdot s_{n+1}^{i_{n+1}}$ is a global section of $L^d$. We will call sections of this forms homogeneous monomials of degree $d$ in the $F_i$’s.

The number $N_d$ of homogeneous polynomials of degree $d$ in the $F_i$’s is $\binom{n+1+d}{n}$, thus there is a positive constant $A$ (independent on $d$) such that $N_d \geq Ad^{n+1}$. Since, by Theorem 2.15, we have that $h^0(X,L^\otimes d) \leq C \otimes d^n$, for $d$ sufficiently big $d$ we have $N_d \geq h^0(X,L^\otimes d)$. Consequently, for $d$ big enough, the set of homogeneous monomials of degree $d$ in the $F_i$ is $C$- linearly dependent.

Suppose that $\sum_{I \in \mathbb{N}^{n+2}}, |I| = d a_I s^I = 0$ be linear dependence relation within the homogeneous monomials of degree $d$ in the $F_i$’s and let $P(X_0,\ldots,X_{n+1})$ be the polynomial $\sum_{I \in \mathbb{N}^{n+2}}, |I| = d a_I X^I \in \mathbb{C}[X_0,\ldots,X_{n+1}]$. Then $P(1,F_1,\ldots,F_{n+1}) = 0$ on $U$. Since $U$ is dense in $X$, the conclusion follows.

In order to prove Theorem 2.15 we begin by recalling the classical Schwartz Lemma:

**Lemma 2.16.** Let $f(z_1,\ldots,z_n)$ be a holomorphic function on the $n$- dimensional disk $B(0,r+\epsilon)$. Suppose that the order of vanishing of $f$ at the origin is at least $d$ and denote by $M := \max_{z \in B(0,r)} \{|f(z)|\}$. Then, for every $z_0$ the disk $B(0,r)$ we have

$$|f(z)| \leq M \cdot r^d. \quad (2.7)$$

We recall that the fact that $f(z)$ has order of vanishing at least $d$ at the origin means that the Taylor expansion of $f$ at the origin is

$$f(z_1,\ldots,z_n) = \sum_{|I| \geq d} a_I z^I. \quad (2.8)$$

The one dimensional case of Lemma 2.16 can be found in any classical text of complex analysis and the multidimensional case can be easily reduced to it.

We can now give the proof of Theorem 2.15

**Proof. (of Theorem 2.15)** Since $X$ is compact, we can find a finite set of points $a_i \in X$ and open neighbourhoods $U_i$ of them with the following properties:

- Each $U_i$ is biholomorphic to the open ball $B(0,1)$ centered in $a_i$.
- Denote by $V_i$ the open balls centered in $a_i$ and with radius $\exp(-1)$. We suppose that the open sets $V_i$ also cover $X$.
- The restriction of the line bundle $L$ to $U_i$ is trivial and we denote by $g_{ij}$ the transition functions of $L$ with respect to the covering given by $U_i$. 


Observe that the \( g_{ij} \) give also the transition functions of \( L \) with respect to the covering given by the \( V_i \).

Denote by \( B \) the real number \( \max_{ij} \{ \log |g_{ij}| \} \) and by \( A \) the smallest integer strictly bigger then \( B \).

We claim the following Lemma:

**Lemma 2.17.** Suppose that \( s \in H^0(X,L^\otimes d) \) is a global section which vanishes at order at least \( d \cdot A + 1 \) at every \( a_i \), then \( s = 0 \).

**Proof. (Of the Lemma)** Let \( s \) be such a section. We can write \( s|_{U_i} = f_i \) where \( f_i \) are holomorphic functions on \( B(0,1) \) and \( f_i = g_{ij}^d \cdot f_j \). Since \( X \) is compact, there exists \( s_0 \) and \( z_0 \in V_{s_0} \) such that \( M := |f_{s_0}(z_0)| = \max_j \{|f_j(z)| / z \in V_j \} \).

Observe that \( z_0 \) cannot be in \( V_{s_0} \cup_{j \neq s_0} V_j \). Indeed \( f_{s_0}(z) \) can only reach his maximum on \( V_{s_0} \) on the border of it. By Schwartz lemma 2.16 we obtain then:

\[
M = |f_{s_0}(z_0)| = |g_{s_0j}|^d \cdot |f_j(z_0)| \leq \exp(d \cdot A) \cdot M \exp(-d \cdot A - 1)
\]

which gives \( M = 0 \). The conclusion of the Lemma follows. \( \Box \)

Let’s show how Lemma 2.16 implies Theorem 2.15:

Let \( I_{a_i} \) be the ideal sheaf of the points \( a_i \). Let \( A \) be as in Lemma 2.16. We have a natural linear map

\[
\alpha_d : H^0(X,L^\otimes d) \longrightarrow \bigoplus_i L^\otimes d \otimes \mathcal{O}_X/I_{a_i}^{d \cdot A + 1}
\]

Lemma 2.16 tells us that for every \( d \), the map \( \alpha_d \) is injective. Indeed a section in the kernel of it would be a section which vanishes at order at least \( d \cdot A + 1 \) at every \( a_i \).

For every positive integer \( d \), we have that \( \mathcal{O}_X/I_{a_i}^d \simeq \mathbb{C}[x_1, \ldots, x_n]/(x_1^d, \ldots, x_n) \). And this last one is a \( \mathbb{C} \) vector space of dimension \( \binom{d+n}{n} \sim d^n \). The conclusion of the proof follows. \( \Box \)

### 3. The Theorem of Chow on analytic subvarieties of the projective space

In this section we will prove that Theorem 2.15 implies also the Chow Theorem:

**Theorem 3.1.** Let \( X \) be a smooth closed analytic of the projective space \( \mathbb{P}^N \). Then \( X \) is algebraic.

This theorem tells us that if a smooth proper analytic variety \( X \) is contained in the projective space (or more generally, inside any projective variety) then it is actually defined by algebraic equations.

As a easy corollary we obtain:

**Corollary 3.2.** Let \( f : X \to Y \) be an holomorphic morphism between smooth projective varieties. Then \( f \) is actually an algebraic map.
Proof. ((Of the Corollary) It suffices to apply Theorem 3.1 to the graph of \( f \) inside \( X \times Y \). This graph will be an algebraic subvariety of \( X \times Y \) thus the morphism \( f \) which is given by the projection on \( Y \) is algebraic. \( \square \)

Another consequence is:

**Corollary 3.3.** Let \( X \) be a smooth projective variety. Then the field of meromorphic functions \( \mathcal{M}_X \) coincides with the field of rational functions \( \mathbb{C}(X) \).

*Proof. (Of the Corollary)* Each meromorphic function corresponds to an holomorphic morphism \( f : X \to \mathbb{P}^1 \). Then apply the Corollary above. \( \square \)

*Proof. (Of Theorem 3.1)* Let \( \overline{X} \) be the Zariski closure of \( X \) inside \( \mathbb{P}^N \). By construction, \( \overline{X} \) is the smallest algebraic subvariety of \( \mathbb{P}^N \) containing \( X \). It suffices to prove that the dimension of \( \overline{X} \) is the same of the dimension of \( X \).

Indeed \( \overline{X} \) is an algebraic variety and in particular an analytic variety. And it is easy to see that an analytic variety contains only one closed subvariety of the same dimension of it.

Let \( n \) be the dimension of \( X \) and \( m \) be the dimension of \( \overline{X} \). A priori we have \( m \geq n \).

Let \( \mathcal{O}(1) \) be the tautological line bundle of \( \mathcal{P}^N \). We first have the following Lemma:

**Lemma 3.4.** Let \( Y \) be a projective variety of dimension \( m \) in \( \mathbb{P}^N \). Then there is a constant \( C \) such that, for every positive integer \( d \), we have

\[
(3.1) \quad h^0(Y, \mathcal{O}(d)) \geq C \cdot d^m.
\]

*Proof. (of the Lemma)* Choose a suitable projection \( h : Y \to \mathbb{P}^m \) which is surjective and generically finite. Then we have an injective linear map

\[
(3.2) \quad h^* : H^0(\mathbb{P}^m; \mathcal{O}(d)) \to H^0(Y, \mathcal{O}(d)).
\]

Since \( h^0(\mathbb{P}^m; \mathcal{O}(d)) \sim d^m \), the conclusion of the Lemma follows. \( \square \)

Since \( X \) is Zariski dense in \( \overline{X} \), we have a natural inclusion

\[
(3.3) \quad H^0(\overline{X}, \mathcal{O}(d)) \hookrightarrow H^0(X, \mathcal{O}(d)|_X).
\]

Consequently, an application of Theorem 2.15 and Lemma 3.4 gives that we can find positive constants \( C_1 \) and \( C_2 \) such that, for every \( d \gg 0 \), we have

\[
(3.4) \quad C_1 \cdot d^m \leq h^0(\overline{X}, \mathcal{O}(d)) \leq h^0(X, \mathcal{O}(d)|_X) \leq C_2 \cdot d^m.
\]

Which implies that \( m \leq n \). Thus \( m = n \) and the conclusion of Theorem 3.1 follows. \( \square \)
4. The Andreotti – Frankel approach to Lefschetz’s Theorem of hyperplane sections

In this section we will explain the main steps of the proof of Lefschetz’s Theorem of fundamental group $\pi_1$ of Hyperplane sections of a projective variety.

Let’s begin by stating the theorem: We will state it only for the first homotopy group but the general statement concerns also some higher homotopy groups:

**Theorem 4.1.** Suppose that $X$ is a smooth projective variety of dimension $n$ at least three and $i : D \hookrightarrow X$ is a smooth ample divisor, then the natural map

$$
i_* : \pi_1(D) \longrightarrow \pi_1(X)$$

is an isomorphism.

There exists another theorem on hyperplane sections which concerns cohomology with values in $\mathbb{Z}$. Since it is not strictly related with the topic of these lectures, we prefer not to quote it. We would like just to mention that the proof of it follows the same lines of the proof of Theorem 4.1.

The proof rely on basic Morse theory.

**Remark 4.2.** Observe that the hypothesis on the dimension is necessary: If you take $X = \mathbb{P}^2$ and $D$ be a smooth projective curve of degree at least three, then $X$ is simply connected and $D$ is not.

In order to prove Theorem 4.1 we need to recall the main Theorem of Milnor Theory.

Let $M$ be a smooth differentiable variety of dimension $r$ and

$$f : M \longrightarrow \mathbb{R}$$

be a smooth function. A point $p \in M$ is said to be critical if $df_p = 0$.

Let $p_0 \in M$ be a critical point for the map $f$. We may define a symmetric bilinear $Hess_f(p_0)$, called the Hessian of $f$ form on the tangent space $T_{p_0}M$ in the following way:

Let $v$ and $w$ be two tangent vectors to $M$ at $P_0$. Extend them to vector fields $\tilde{v}$ and $\tilde{w}$ in a neighbourhood of $p_0$. Then we define

$$Hess_f(p_0)(v, w) := v(\tilde{w}(f)).$$

It is symmetric and well defined because $v(\tilde{w}(f))(p_0) - w(\tilde{v}(f))(p_0) = [\tilde{v}, \tilde{w}](f)(p_0) = 0$ because $p_0$ is degenerate.

In local coordinates: if $(x_1, \ldots, x_r)$ are local coordinates of $M$ around $p_0$ and $f$ is given by a smooth function $f(x_1, \ldots, x_r)$ then, a local computation shows that $Hess_f(p_0)$ is the quadratic form associated to the symmetric matrix

$$\begin{pmatrix}
\frac{\partial^2 f}{\partial x_i \partial x_j}
\end{pmatrix}$$
with respect to the basis \( \left( \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_r} \right) \). The point \( p_0 \) is said to be \textit{critical non degenerate} if the quadratic form \( \text{Hess}_f(p_0) \) is non singular.

Suppose that \( p_0 \) is a non degenerate critical point for \( f \), Inside \( T_{p_0}M \) there is a biggest subspace \( V \) where the quadratic form \( \text{Hess}_f(p_0) \) is \textit{negative defined}. We will call the dimension of \( V \) the \textit{index} of \( f \) at \( p_0 \).

Remark 4.3. Observe that, by definition, the fact that a point is critical for a map \( f \) or that it is non degenerate or its index is independent on the choice of local coordinates around it.

Given a smooth function \( f : M \to \mathbb{R} \) and \( a \in \mathbb{R} \), we will denote by \( M_a \) the closed set \( f^{-1}((-\infty, a]) \).

The main Theorem of Morse theory is the following:

**Theorem 4.4.** Let \( M \) be a differential variety and \( f : M \to \mathbb{R} \) be a differential map. Suppose that for every \( a \in \mathbb{R} \) the subset \( M_a \). Let \( c \in \mathbb{R} \) and suppose that \( f^{-1}(c) \) contains \( k \) non degenerate critical points \( p_1, \ldots, p_k \) of index \( \lambda_1, \ldots, \lambda_k \) respectively and no other critical points. Suppose that, for \( \epsilon > 0 \), the subset \( M_{c+\epsilon} \) do not contain any other critical points. Then \( M_{c+\epsilon} \) has the homotopy type of \( M_{c-\epsilon} \) with a cell of dimension \( \lambda_i \) attached to each point \( p_i \).

We recall that a cell of dimension \( \lambda \) is just the space \( \sigma_\lambda := \{(x_1, \ldots, x_\lambda) \in \mathbb{R}^\lambda / \sum x_i^2 \leq 1 \} \) and its border is \( S_\lambda := \{(x_1, \ldots, x_\lambda) \in \mathbb{R}^\lambda / \sum x_i^2 = 1 \} \). To attach a cell to a space means to attach \( \sigma_\lambda \) ”along” \( S_\lambda \).

Another important Lemma we need is the following:

**Theorem 4.5.** Let \( M \) be a smooth closed manifold contained in \( \mathbb{R}^n \). For every \( p \in \mathbb{R}^n \) define the map

\[
(4.5) \quad d_p : M \longrightarrow \mathbb{R} \quad \text{ such that } \quad x \longmapsto d_p(x) := ||x - p||^2
\]

Then, for almost all \( p \in \mathbb{R}^n \), the map \( d_p \) has only non degenerate critical points.

The first consequence of Theorems 4.4 and 4.5 is:

**Corollary 4.6.** Each smooth projective variety is homotopy equivalent to a CW complex.

The proof of the corollary can be done by induction on the dimension of the variety and it is left by exercise.

Let’s start the proof of Theorem 4.1.

**Proof.** (Of Theorem 4.1) The first part of the proof relies on the following Theorem.

**Theorem 4.7.** Let \( A \subset \mathbb{C}^n \) be a smooth affine variety of dimension \( r \). Identify \( \mathbb{C}^n \cong \mathbb{R}^{2n} \). For \( p \in \mathbb{R}^{2n} \) generic, consider the map \( d_p : A \to \mathbb{R} \). Suppose that \( x_0 \in A \) is a non degenerate critical point of \( f_p \). Then the index of \( x_0 \) with respect to \( f_p \) is at most \( r \).
Suppose that $A$ is a closed manifold of $\mathbb{R}^n$ and $x_0 \in A$ is a non degenerate critical point of index $\lambda$ for a map $d_p : A \to \mathbb{R}$. Let $H \in O(n)$ be an isometry, $b \in \mathbb{R}^n$ and $a \in \mathbb{R}_{>0}$. Then, a direct computation shows that $x_0$ is a non degenerate critical point for the map $d_p(a \cdot H(x) + b)$.

Consequently we may suppose:

i) $x_0 = (0, \ldots, 0)$;

ii) the point $p = (0, \ldots, 1, 0, \ldots, 0)$ where the 1 is at the $(r + 1)$'s position.

iii) The variety $A$ is given by the graph of a holomorphic function $(f_{r+1}(z), \ldots, f_n(z)) : \mathbb{C}^r \to \mathbb{C}^{n-r}$.

Moreover, the map $d_p$ is given by

$$d_p(z_1, \ldots, z_r) = \sum_{i=1}^r |z_i|^2 + |f_{r+1}(z) - 1|^2 + \sum_{j=r+2}^n |f_j(z)|^2.$$  

(4.6)

And, since $x_0$ is critical for $d_p$, a direct computation gives that each $f_i$ will have order of vanishing at least two at the origin. We can also rewrite the function $d_p(z_1, \ldots, z_r)$ as

$$d_p(z_1, \ldots, z_r) = (1 - 2 \cdot \text{Re}(f_{r+1}(z))) + \sum_{i=1}^r |z_i|^2 + \sum_{j=r+2}^n |f_j(z)|^2.$$  

(4.7)

Since each of the $i(z)$'s has has order of vanishing at least two at the origin, we see that they not contribute to the Hessian of $d_p$ at $x_0$. Consequently, if write the Taylor expansion of $f_{r+1}(z_1, \ldots, z_r) = F_2(z) + F(z)$ where $F_2(z)$ is an homogeneous polynomial of degree two in the $z_i$'s and $F(z)$ is a holomorphic function with order of vanishing at least three at the origin, we obtain:

$$Hess_{d_p}(x_0) = 2(-\text{Hess}_{\text{Re}(F_2(z))}(x_0) + \text{Id}_r).$$  

(4.8)

From equation 4.8 above we see that if $\lambda$ is a negative eigenvalue of $Hess_{d_p}(x_0)$, then $\frac{1-\lambda}{2}$ is a positive eigenvalue of $Hess_{\text{Re}(F_2(z))}(x_0)$. Consequently Theorem 4.7 is a consequence of the following Lemma:

**Lemma 4.8.** Let $F_2(z_1, \ldots, z_r)$ be a homogeneous polynomial of degree two. Then $\text{Hess}_{\text{Re}(F_2(z))}(0)$ has at most $r$ positive eigenvalues.

**Proof.** It suffices to remark that, after a change of complex variables, we may suppose that

$$F_2(z_1, \ldots, z_r) = z_1^2 + \cdots + z_s^2$$  

(4.9)

with $s \leq r$. Consequently, if we write $z_i = x_i + \sqrt{-1}y_i$ we obtain

$$Hess_{\text{Re}(F_2(z))}(0) = x_1^2 - y_1^2 + \cdots + x_s^2 - y_s^2.$$  

(4.10)

\[ \square \]

We now come to the last part of the proof of Theorem 4.1. We may suppose that $X$ is embedded in a projective space, $D$ is the hyperplane at infinity of $X$ and $A = X \setminus D$ is a smooth affine variety embedded in the corresponding affine space.
Remark that a priori, $D$ may be only ample and not very ample, in this case the hyperplane at infinity will be $D$ with some multiplicity but this will not affect the fundamental groups.

Since $D$ and $X$ are homotopically equivalent to CW complexes, a small neighbourhood $V$ of $D$ in $X$ can be deformed to $D$ inside it.

We consider the function $f : X \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x \in D \\ \frac{1}{d_p(x)} & \text{if } x \in A \end{cases}$$

It is easy to see that, since, by Theorem 4.7, the critical points of $d_p(\cdot)$ have are non degenerate with index less or equal to $r$, the critical points of $f(x)$ inside $A$ will be non degenerate of index bigger then $2r - 2 = r$. Consequently, for every $\epsilon > 0$ sufficiently small, $X$ is homotopically equivalent to $X_\epsilon$ with cells of dimension at least $r$ attached. Consequently, since any compact complex variety is homotopically equivalent to a CW complex and the fundamental group of a CW complex do not change if we add to it cells of dimension at least three, the natural map

$$\pi_1(X_\epsilon) \longrightarrow \pi_1(X)$$

is an isomorphism.

We may suppose that $X_\epsilon \subset V$. Observe that $D \subset X_\epsilon$. Consequently we have natural maps

$$\pi_1(D) \xrightarrow{\alpha} \pi_1(X_\epsilon) \xrightarrow{\beta} \pi_1(V)$$

and

$$\pi_1(X_\epsilon) \xrightarrow{\gamma} \pi_1(V) \xrightarrow{\delta} \pi_1(X)$$

Since, $\beta \circ \alpha$ and $\delta \circ \gamma$ are isomorphisms, the map

$$\pi_1(D) \longrightarrow \pi_1(X)$$

being injective and surjective, is a bijection.

\[\square\]

5. Introduction to formal geometry

In this section all the rings and schemes will be Noetherian and all the morphisms of schemes will be separated.

In this section we will see how we can introduce, in algebraic geometry, some tools which mimic the use of topology and analytic geometry. In some cases they can be used to find algebraic analogues of the tools introduced in the previous sections.

The tools we are speaking about are called "formal geometry" and the starting point is the analogous of the Taylor expansion of a holomorphic function.

The basics of the theory require some commutative algebra. Consequently we will just recall them without proofs:
a) Let $A$ be a ring and $\mathfrak{a}$ be an ideal of it. For every $n$ we have a surjective map $A/\mathfrak{a}^{n+1} \to A/\mathfrak{a}^n$. Let $\hat{A}$ the projective limit $\varprojlim A/\mathfrak{a}^n$ is called the $\mathfrak{a}$–adic completion of $A$. We have a natural map $A \to \hat{A}$. The ring $A$ is said to be $\mathfrak{a}$–adically complete if this map is an isomorphism.

b) More generally, given a $A$–module $M$, we can introduce the $\mathfrak{a}$–adic completion $\hat{M}$ of it in the same way. If $M$ is finitely generated then $\hat{M} = M \otimes_A \hat{A}$.

c) Let $A$ be a ring and $\mathfrak{a}$ be an ideal of it. Suppose that $A$ is $\mathfrak{a}$–adically complete. We associate to the couple $(A, \mathfrak{a})$ a locally ringed space $\text{Spf}(A) := (X_0, \mathcal{O}_{X_0})$ called the formal spectrum of it and defined in the following way:

- The underlying topological space is the topological space $X_0 := \text{Spec}(A/\mathfrak{a})$ with its Zariski topology.
- Denote by $X$ the scheme $\text{Spec}(A)$ and by $\mathcal{I}_\mathfrak{a}$ the sheaf of ideals associated to $\mathfrak{a}$. The scheme $X_0$ is the closed set $V(\mathfrak{a})$ of $X$. For every positive integer $n$ the sheaf $\mathcal{O}_n := \mathcal{O}_X/\mathcal{I}_n$ is supported on $X_0$. The structure sheaf $\mathcal{O}_{X_0}$ is defined then to be the sheaf $\varprojlim \mathcal{O}_n$.

Example 5.1. 1) Let $A = k[[t]]$ where $k$ is a field and $\mathfrak{a} = (0)$. Then $\hat{A} = k[[t]]$ and $\text{Spf}(\hat{A})$ is the ringed space whose underlying topological space is a point and the structure sheaf is $k[[t]]$.

2) Suppose that $A = \mathbb{Z}$ and $\mathfrak{a} = (p)$ is a prime ideal. Then $\text{Spf}(\hat{A})$ is a ringed space whose underlying topological space is a point and the structure sheaf is $\mathbb{Z}_p$ (the ring of $p$–adic numbers).

3) Let $\ell \subset \mathbb{A}^2$ be the line $x = 0$ (the coordinates on $\mathbb{A}^2$ will be $(x, y)$). Let $(x) \subset k[x, y] = A$ be the associated ideal. Then $\text{Spf}(\hat{A})$ is the ringed space whose underlying topological space is $\ell$ and the structure sheaf is the sheaf associated to the ring $k[y][[x]]$.

d) More generally, a locally ringed space $(\mathcal{X}, \mathcal{O}_{\mathcal{X}})$ is said to be a formal scheme if it is locally isomorphic to the formal spectrum of a couple $(A, \mathfrak{a})$.

e) A morphism of formal schemes is a morphism of locally ringed spaces.

f) The dimension of a formal scheme is its Krull dimension. Observe that in general it is bigger then the Krull dimension of the underlying topological space: for instance in the examples (1) and (2) above the formal spectrum has dimension one but the underlying topological space has dimension zero.

g) Given a formal scheme $\mathcal{X}$ there exists an ideal sheaf $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ such that:

- The support of $\mathcal{O}_{\mathcal{X}}/\mathcal{I}$ is the underlying topological $X_0$ space of $\mathcal{X}$;
- The couple $(X_0, \mathcal{O}_{\mathcal{X}}/\mathcal{I})$ is a noetherian scheme.

Such a ideal sheaf is called ideal of definition of the formal scheme $\mathcal{X}$ and the scheme $X_0$ is called support scheme of $\mathcal{X}$.

h) A basic example of formal scheme, which generalises the construction of the formal spectrum is the following: Let $X$ be a scheme and $Y$ be a closed subscheme of it. Denote by $I_Y$ the ideal sheaf of $Y$. The formal completion of $X$ around $Y$ is the formal scheme whose underlying topological space is $Y$ and the structural sheaf is $\varprojlim \mathcal{O}_X/I^n_Y$ and it is denoted by $\hat{X}_Y$. 

The formal completion of $X$ around $Y$ can be imagined as a tubular neighbourhood of $Y$ around $X$.

i) A formal scheme $X$ is said to be algebraizable if there exists a scheme $X$ and a closed subscheme $Y$ such that $X \simeq \hat{X}_Y$. Similarly, a morphism between algebraizable formal schemes $\hat{f} : \hat{X}_Y \to \hat{Z}_T$ is said do be algebraizable if there exists a morphism of schemes $f : X \to Z$ such that the formal completion of it is $\hat{f}$.

j) If $X$ is a scheme and we take $Y = X$, then the formal completion $\hat{X}_Y$ is just $X$. Consequently the category of formal schemes contains the category of schemes.

k) Let $X$ be a formal scheme, $X_0$ be its underlying scheme and $I$ be an ideal of definition of $X$. For every positive integer $n$ the scheme $(X_0, O_{X}/I^n)$ is a closed subscheme of $X$. We will denote it by $X_n$. Consequently we may imagine $X$ as “filtered” by schemes having all the same underlying spaces but structure sheaf containing nilpotents. We have inclusions of schemes $\iota_n : X_n \hookrightarrow X_n+1$.

i) Not every formal morphism is algebraizable: for instance take $(X,Y) = (A^1,0)$ and $(Z,T) = (A^2,(0,0))$. The map associated to the morphism $h : k[[z,w]] \to k[[x]]$ given by $h(z) = x$ and $h(w) = \sum_{n=1}^{\infty} x^n/n!$ is not algebraizable.

We will now describe coherent sheaves on formal schemes:

**Theorem 5.2.** Let $\mathcal{X}$ be a formal scheme and $I$ be an ideal of definition of it.

If $F$ is a coherent sheaf on $\mathcal{X}$ then, for every $n$, the restriction $F_n$ of it to $X_n$ is a coherent sheaf on $X_n$.

Conversely, if for every $n$, we have a coherent sheaf $F_n$ on $X_n$ with exact sequences

\[(5.1) 0 \to I^n \cdot F_{n+1} \to F_{n+1} \to (\iota_n)_*(F_n) \to 0.\]

Then $\lim_\leftarrow F_n$ is a coherent sheaf on $\mathcal{X}$.

Observe that, the theorem above tells us that, if $F$ is a coherent sheaf on $\mathcal{X}_n$ then we have an exact sequence of coherent sheaves on $\mathcal{X}$:

\[(5.2) 0 \to I^n \cdot F \to F \to (\iota_n)_*(\iota_n^*(F)) \to 0.\]

m) A standard fact in commutative algebra is that if $A$ is a noetherian ring and $I$ is an ideal, then the completion $\hat{A}_I$ of $A$ with respect to $I$ is flat over $A$. Consequently we get

**Proposition 5.3.** Let $X$ be a scheme and $Y$ be a closed subscheme of it. Then the natural morphism $\iota : \hat{X}_Y \to X$ is a flat morphism.

In particular we obtain:

**Proposition 5.4.** In the hypothesis of Proposition 5.3, the functor $\iota^* : \text{Coh}(X) \to \text{Coh}(\hat{X}_Y)$ ($\text{Coh}(\cdot)$ being the category of coherent sheaves) is exact.

Another important tool in formal geometry is the so called *Theorem of Formal functions*: 

We recall the following general fact on higher direct images of coherent sheaves: Suppose we have a cartesian diagram of schemes:

\[
\begin{array}{ccc}
X & \xrightarrow{\iota_X} & X \\
g & \downarrow & f \\
T & \xrightarrow{\iota_Y} & Y
\end{array}
\]

and a coherent sheaf \(F\) on \(X\). Then, for every positive integer \(i\), there is a natural morphism of coherent sheaves on \(T\)

\[
i^* R^i f_*(F) \longrightarrow R^i g_*(\iota_X^*(F)).
\]

We suppose that we have a proper morphism of schemes \(f : X \rightarrow Y\). Let \(S\) be a proper closed subset of \(Y\) and \(F\) be a coherent sheaf on \(X\). Denote by \(T = f^{-1}(S)\).

Let \(\hat{X}_T\) and \(\hat{Y}_S\) be the completions of \(X\) (resp. \(Y\)) with respect to \(T\) (resp. \(S\)). We have a cartesian diagram

\[
\begin{array}{ccc}
\hat{X}_T & \xrightarrow{\iota_X} & X \\
j & \downarrow & f \\
\hat{Y}_S & \xrightarrow{\iota_Y} & Y.
\end{array}
\]

From Proposition 5.4 we get:

**Proposition 5.5.** Let \(F\) be a coherent sheaf on \(X\), then, for every integer \(i\), the natural map

\[
(t_Y)^*(R^i f_*(F)) \longrightarrow R^i \hat{f}_*(\iota_X^*(F))
\]

is an isomorphism.

For every positive integer we may consider the \(n\)-th infinitesimal neighbourhoods \(S_n\) and \(T_n\) of \(S\) and \(T\) in \(Y\) and \(X\) respectively: If \(I_S\) (resp. \(I_T = I_S \cdot \mathcal{O}_X\)) is the ideal sheaf of \(S\) in \(Y\) (resp. of \(T\) in \(X\)), then \(S_n\) (resp. \(T_n\)) is the closed subset associated to the ideal sheaf \(I^n_S\) (resp. to the ideal \(I^n_T\)). Then, for every \(n\), we have a cartesian diagram:

\[
\begin{array}{ccc}
T_n & \xrightarrow{\iota_n} & X \\
f_n & \downarrow & f \\
S_n & \xrightarrow{j_n} & Y.
\end{array}
\]

which factorises:

\[
\begin{array}{ccc}
T_n & \xrightarrow{h_n} & \hat{X}_T & \xrightarrow{\iota_X} & X \\
f_n & \downarrow & f & \downarrow & f \\
S_n & \xrightarrow{g_n} & \hat{Y}_S & \xrightarrow{\iota_Y} & Y.
\end{array}
\]
Let $F$ be a coherent sheaf on $X$. From the cartesian diagram above we get, for every $i$ and $n$, morphisms:

$$R^i\hat{f}_*(\iota_X^*(F)) \sim \iota_Y^*(R^i f_*(F)) \to (g_n)_*(j_n^*(R^i f_*(F))) = (g_n)_*(j_n^*(R^i f_*(F)))$$

We also have natural morphisms

$$(j_n)^*(R^i f_*(F)) \to R^i(f_n)_*(\iota_Y^*(F))$$

Consequently we get, for every $i$ and $n$, natural maps

$$R^i\hat{f}_*(\iota_X^*(F)) \to (g_n)_*(R^i(f_n)_*(F))$$

which eventually gives a map

$$R^i\hat{f}_*(\iota_X^*(F)) \to \varprojlim (g_n)_*(R^i(f_n)_*(F)).$$

The Theorem of Formal functions is the following:

**Theorem 5.6. (Theorem of Formal Functions) If the morphism $f : X \to Y$ is proper, then the morphism in 5.12 is an isomorphism.**

### 5.1. The Grothendieck algebraization Theorem

We will now prove a theorem which is the analogue of the Chow Theorem on analytic subvarieties of the projective space in the contest of formal geometry. This theorem is usually applied in deformation theory.

We begin with a Theorem which is an analogue of Serre G.A.G.A. in formal geometry. It is often called Formal G.A.G.A.

We suppose that $A$ and $I$ is an ideal of $A$. We suppose that $A$ is complete with respect to the $I$–adic topology: the natural morphism $A \to \hat{A} := \varprojlim A/I^n$ is an isomorphism. If $f : X \to \text{Spec}(A)$ is a $A$–scheme, for every positive integer $n$, consider the cartesian diagram

$$X_n := X \times_A \text{Spec}(A/I^n) \to X$$

Thus we get a formal scheme $\hat{X} := \varprojlim X_n$ and a cartesian diagram

$$\hat{X} \overset{i_X}{\to} X$$

**Theorem 5.7. (Grothendieck Existence Theorem) Suppose, under the hypothesis above, that the morphism $f : X \to \text{Spec}(A)$ is proper, then the natural functor**

$$\iota_X^*(\cdot) : \text{Coh}(X) \to \text{Coh}(\hat{X})$$
is an equivalence of categories.

Proof. We will prove the Theorem under the hypothesis that $f$ is projective.

We begin by proving that $i^*_X(\cdot)$ is fully faithful: Let $F$ and $G$ be two coherent sheaves on $X$. We want to prove that the natural morphism

$$\text{Hom}(F, G) \rightarrow \text{Hom}(i^*_X(F); i^*_X(G))$$

is an isomorphism.

- By flatness, we have

$$j^*(f_*(\mathcal{H}om(F, G))) \simeq \hat{f}_*(i^*_X(\mathcal{H}om(F, G))).$$

Moreover $\mathcal{H}om(i^*_X(F), i^*_X(G)) \simeq i^*_X(\mathcal{H}om(F, G))$. 

- Since $\text{Spec}(A)$ is affine, we have $\text{Hom}(F, G) = f_*(\mathcal{H}om(F, G))$.

- Since $\text{Spf}(A)$ is an affine formal scheme, we have

$$\text{Hom}(\hat{i}^*_X(F), \hat{i}^*_X(G)) = \hat{f}_*((\mathcal{H}om(i^*_X(F), i^*_X(G))))).$$

Consequently, gathering all together, we obtain

$$j^*(\text{Hom}(F, G)) \simeq \text{Hom}(i^*_X(F), i^*_X(G)).$$

- Since $A$ is $I$–adically complete and $\text{Hom}(F, G)$ is of finite type (because $f$ is proper), we have isomorphisms $\text{Hom}(F, G) \simeq \hat{\text{Hom}}(F, G) = j^*(\text{Hom}(F, G))$. The fully faithfulness follows.

In order to conclude, we need to prove that if $\hat{G}$ is a coherent sheaf on $\hat{X}$ then there exists a coherent sheaf $G$ on $X$ such that $\hat{G} \simeq i^*_X(G)$.

Since we are supposing that $f : X \rightarrow \text{Spec}(S)$ is projective, we may suppose the existence of a relatively ample line bundle $\mathcal{O}_X(1)$. We will denote by $\mathcal{O}(1)$ its pull back to $\hat{X}$.

A key step of the proof is the following Lemma:

**Lemma 5.8.** Under the hypothesis of Theorem 5.7, for every coherent sheaf $\hat{G}$ on $\hat{X}$ we can find a surjective map

$$\bigoplus_{i=1}^r \hat{\mathcal{O}}(d_i) \rightarrow \hat{G} \rightarrow 0.$$ 

Let’s show how Lemma 5.8 implies Theorem 5.7: Lemma 5.8 implies that we can find an exact sequence

$$\bigoplus_{j=1}^s \hat{\mathcal{O}}(\ell_j) \xrightarrow{h} \bigoplus_{i=1}^r \hat{\mathcal{O}}(d_i) \rightarrow \hat{G} \rightarrow 0.$$

Since $i^*_X(\cdot)$ is fully faithful, we can find a morphism $h : \bigoplus_{j=1}^s \hat{\mathcal{O}}(\ell_j) \rightarrow \bigoplus_{i=1}^r \hat{\mathcal{O}}(d_i)$ on $X$ such that $i^*_X(h) = \hat{h}$.

Let $G := \text{Coker}(h)$. We have that $i^*_X(G) = \hat{G}$ and the conclusion of the Theorem follows. \qed
Proof. (Of Lemma 5.8) In order to prove that such a surjective map exists, we need the following general Lemma:

**Lemma 5.9.** Let \( f : X \to Y \) be a proper morphisms. Let \( S = \oplus_i S_i \) be a graded \( \mathcal{O}_Y \) algebra generated by \( S_1 \) and let \( M = \oplus_{j \in \mathbb{Z}} M_j \) be a coherent \( f^*(S) \) module of finite type. Then for all integer \( q \) the module

\[
R^q f_*(M) := \oplus_{j \in \mathbb{Z}} R^q f_*(M_j)
\]

is a \( S \)-module of finite type and there exists \( n_0 \) such that, for every \( n \geq n_0 \) we have

\[
R^q f_*(M_n) = S_{n-n_0} \cdot R^q f_*(M_{n_0}).
\]

**Proof. (Sketch)** Define \( \tilde{Y} := \text{Spec}(S) \to Y \) and \( \tilde{X} := \text{Spec}(f^*(S)) \to X \). We have a cartesian diagram

\[
\begin{array}{ccc}
\tilde{X} & \longrightarrow & X \\
\tilde{f} \downarrow & & \downarrow f \\
\tilde{Y} & \longrightarrow & Y
\end{array}
\]

and \( \tilde{f} \) is proper by base change. The \( f^*(S) \) module \( M \) gives rise to a \( \mathcal{O}_{\tilde{X}} \)–module which is coherent because \( M \) is of finite type. Since \( \tilde{f} \) is proper, \( R^q \tilde{f}_*(M) \) is a coherent \( \mathcal{O}_{\tilde{Y}} = S \) module of finite type. Thus the first part of the proposition. The second part is a standard fact on moduli of finite type over graded algebras. \( \square \)

Let’s go back to the proof of the Lemma. Denote by \( \iota_n : X_n \to \tilde{X} \) the inclusion and by \( f_n : X_n \to \text{Spec}(A/I^n) \) the natural map. The morphism \( f_n \) is projective and the line bundle \( \mathcal{O}_{n}(1) := \iota_n^*(\hat{\mathcal{O}}(1)) \) is ample over \( X_n \) (because \( X_n \) is also a subscheme of \( X \)).

Let \( \tilde{G} \) be a coherent module over \( \tilde{X} \). Denote by \( G_n \) its restriction to \( X_n \).

Consider the \( A \) algebra \( S = \oplus_i I^i I^{i+1} \). It is a graded \( A/I \) algebra of finite type generated by \( S_1 \). On \( X_1 \) we define the \( f_1^*(S) \) module of finite type \( M = \oplus_{j \in \mathbb{N}} I^j \cdot G/I^j+1 \cdot G \). Observe that \( M_1 \) is just \( G_1 \).

As in the proof of Lemma 5.9, consider the diagram

\[
\begin{array}{ccc}
\tilde{X}_1 & \longrightarrow & X \\
\tilde{f}_1 \downarrow & & \downarrow f_1 \\
\text{Spec}(A/I) & \longrightarrow & \text{Spec}(A/I).
\end{array}
\]

Since \( \mathcal{O}_{1}(1) \) is ample on \( X_1 \), the line bundle \( g^*(\mathcal{O}_{1}(1)) \) is relatively ample with respect to \( \tilde{f} \). Consequently, we can find a integer \( d_0 \) such that, for every \( d \geq d_0 \) and \( q > 0 \), we have

\[
R^q \tilde{f}_*(M) = 0.
\]

\[
\text{(5.26)}
\]
Which means that we can find a positive integer $d_0$, such that, for every positive integer $d \geq d_0$, for $q > 0$ and for every positive integer $j$, we have

\[ R^q(f_1)_*(I^j \cdot G / I^{j+1} \cdot G \otimes \mathcal{O}_1(d)) = 0. \]  

and since $\text{Spec}(A/I)$ is affine, we get that for every positive integer $d \geq d_0$, for $q > 0$ and for every positive integer $j$, we have

\[ H^q(X_1; I^j \cdot G / I^{j+1} \cdot G \otimes \mathcal{O}_1(d)) = 0. \]

Again by ampleness of $\mathcal{O}_1(1)$ we may suppose that, for $d \geq d_0$ we have that $G_1(d)$ is generated by global sections.

Consider now the exact sequence

\[ 0 \longrightarrow I^j \cdot G / I^{j+1} \cdot G \otimes \mathcal{O}_1(d) \longrightarrow G_{j+1} \longrightarrow G_j \longrightarrow 0 \]

Taking global sections of it we get

\[ 0 \longrightarrow H^0(X_1; I^j \cdot G / I^{j+1} \cdot G \otimes \mathcal{O}_1(d)) \longrightarrow H^0(X_{j+1}; G_{j+1}) \longrightarrow H^0(X_j; G_j) \longrightarrow \]

\[ \longrightarrow H^0(X_j; G_j) \longrightarrow H^1(X_1; I^j \cdot G / I^{j+1} \cdot G \otimes \mathcal{O}_1(d)) \]

But, since we just proved that the last term of this exact sequence vanishes, a direct application of the Theorem of formal functions gives a surjective map

\[ H^0(\hat{X}, \hat{G}(d)) \longrightarrow H^0(X_1, G_1(d)) \longrightarrow 0. \]

Since $G_1(d)$ is generated by global sections and because the surjective morphism above, we have a commutative diagram

Take a finitely generated $A$ module $V$ of global sections of $\hat{G}(d)$ which generate $G_1$ (they exist because $G_1(d)$ is generated by global sections) and we have the surjection above.

\[ V \otimes \mathcal{O}_{\hat{X}} \xrightarrow{\beta} \hat{G}(d) \]

\[ \alpha \]

\[ G_1(d). \]

The map $\alpha$ is surjective and consequently, by Nakayama Lemma, the map $\beta$ is also surjective. This means that $\hat{G}(d)$ is generated by global sections. Consequently the map

\[ V \otimes \hat{\mathcal{O}}(-d) \longrightarrow \hat{G} \]

is surjective. Which ends the proof of the Lemma.

\[ \square \]

Theorem 5.7 and also Lemma 5.8 have many interesting corollaries:

**Corollary 5.10.** Under the hypothesis of Theorem 5.7, every formal closed subscheme of $\hat{X}$ is algebraizable.
Corollary above may be seen as an analogue of Chow Theorem in the contest of formal geometry.

Proof. Let \( \hat{Z} \) be such a scheme and \( I_Z \) be its ideal sheaf. Since \( I_Z \) is coherent, there exists an ideal sheaf \( I_Z \) such that \( i_X(I_Z) = I_{\hat{Z}} \). Consequently \( \hat{Z} \) is the formal completion of \( Z \) and the conclusion follows.

If we apply the corollary to the graph of a formal function, we find

**Corollary 5.11.** Under the hypothesis of Theorem 5.7 suppose that \( X \) and \( Y \) are two schemes proper over \( A \). Let \( \hat{g} : \hat{X} \to \hat{Y} \) be a formal map. Then \( \hat{f} \) is algebraizable: there is an algebraic map \( f : X \to Y \) the completion of which is \( \hat{f} \).

The last consequence we’d like to mention is a refinement of Corollary 5.10 and it can be seen as an "algebraization theorem"

**Theorem 5.12.** Let \( A \) be a \( \mathcal{I} \)-adically complete ring. Suppose that \( \hat{f} : \mathcal{X} \to \text{Spf}(A) \) is a proper formal scheme having as ideal of definition \( \hat{f}^*(I) \). Suppose that \( \mathcal{L} \) is a line bundle on \( \mathcal{L} \) the restriction of which to \( X_1 \) is ample. Then \( \mathcal{X} \) is algebraizable.

**Proof.** As in the proof of Lemma 5.8, we can find a finitely generated free \( A \)-module \( V \) and a surjective map,

\[
V \otimes \mathcal{O}_X \longrightarrow L^\otimes d
\]

for \( d \) big enough. Suppose that the rank of \( V \) is \( N + 1 \) and let \( \hat{\mathbb{P}}^N_A \) be the completion of \( h : \mathbb{P}_A^N \to \text{Spec}(A) \) along \( h^*(I) \). For every \( n \), the surjection above give rise to an closed inclusion of \( X_n \) inside \( \mathbb{P}_n^N \). Consequently, \( \mathcal{X} \) can be realized as a closed formal subscheme of \( \hat{\mathbb{P}}^N_A \). The conclusion follows from Corollary 5.10.

**5.2. A formal scheme which is not algebraizable.** This example is taken from [3] (but described in a slightly different way). We will now describe an example of formal scheme which is not algebraizable. We recall a formal scheme \( \mathcal{X} \) is algebraizable if there is a scheme \( X \) and closed subscheme \( Y \) such that \( \mathcal{X} \simeq \hat{X}_Y \).

In this part we will suppose that the underlying field is algebraically closed of characteristic zero (this hypothesis is not necessary, but it will simplify the proof).

Before we describe the example, we need to recall some facts of algebraic geometry which will be needed.

**5.2.1. Étale covering of formal schemes.** Let \( X \) be a scheme. Suppose that there exists an ideal sheaf \( J \subset \mathcal{O}_X \) such that \( J^2 = 0 \). Let \( X_0 \) be the closed subscheme of \( X \) associated to \( J \).
Suppose that \( f_0 : Y_0 \to X_0 \) is a finite étale covering of \( X_0 \). Then we can find a unique finite étale covering \( f : Y \to X \) "extending" \( f_0 \). This means that there exists a cartesian diagram

\[
\begin{array}{ccc}
Y_0 & \longrightarrow & Y \\
\downarrow f_0 & & \downarrow f_1 \\
X_0 & \longrightarrow & X.
\end{array}
\]

This easily implies

**Theorem 5.13.** Let \( \mathcal{X} \) be a formal scheme with ideal of definition \( \mathcal{I} \) and \( X_0 \) support scheme. Then every finite étale covering \( f_0 : Y_0 \to X_0 \) extends to a finite étale covering \( f : \mathcal{Y} \to \mathcal{X} : 
\]

\[
\begin{array}{ccc}
Y_0 & \longrightarrow & \mathcal{Y} \\
\downarrow f_0 & & \downarrow f_1 \\
X_0 & \longrightarrow & \mathcal{X}.
\end{array}
\]

is cartesian. Consequently if \( \text{Et}(\mathcal{X}) \) (resp. \( \text{Et}(X_0) \)) is the category of the finite étale covering of \( \mathcal{X} \) (resp. of \( X_0 \)), the natural restriction functor \( \text{Et}(\mathcal{X}) \to \text{Et}(X_0) \) is an equivalence of categories.

The second fact we need is the:

5.2.2. **Nagata compactification Theorem.** We know that an affine variety can be seen as an open set of a projective variety. Nagata compactification Theorem is a generalisation of this fact to any scheme. We state it in the particular case of schemes over a field.

**Theorem 5.14.** (Nagata Compactification Theorem) Let \( X \) be a scheme of finite type over a field. Then there exists a proper scheme \( \overline{X} \) and a open inclusion \( X \hookrightarrow \overline{X} \).

The third fact is:

5.2.3. **Kawamata-Vieweg Vanishing Theorem.** We will need the following:

**Theorem 5.15.** Let \( X \) be a smooth projective surface over a field of characteristic zero. Let \( L \) be a nef and big line bundle on it. Then

\[
H^1(X, L^{-1}) = \{0\}.
\]

We recall that a line bundle \( L \) on a surface \( X \) is said to be nef if, for every curve \( C \) on \( X \), we have \( \deg(L|_C) \geq 0 \). The line bundle \( L \) is said to be \( h^0(X, L^n) \sim n^2 \).
5.2.4. The example. We now describe the example:

Let $C \hookrightarrow \mathbb{P}^2$ be a smooth projective curve of positive genus and degree $d > 1$. Let $g : D \to C$ be a non trivial étale covering.

Let $\hat{\mathbb{P}}_C^2$ be the completion of $\mathbb{P}^2$ along $C$.

By Theorem 5.13 we can find a formal scheme $\mathcal{X}$ and an étale covering $\hat{g} : \mathcal{X} \to \hat{\mathbb{P}}_C^2$ extending the morphism $g$.

We now prove that $\mathcal{X}$ is not algebraizable:

Suppose it is algebraizable, then there is a scheme of dimension two $X$ with a Weil divisor $D$ such that $X \cong \hat{X}_D$.

– Since $\mathcal{X}$ is regular, $X$ must be regular in a neighbourhood of $D$. Thus $D$ is a Cartier divisor on it.

– Using Nagata compactification Theorem 5.14, we may suppose that $X$ is proper.

– The normal bundle of $C$ in $\mathbb{P}^2$, and consequently in $\hat{\mathbb{P}}_C^2$ is $\mathcal{O}(d)|_C$.

– Since $\hat{g}$ is étale, the normal bundle of $D$ in $\mathcal{X}$, and consequently in $X$, is $g^*(\mathcal{O}(d)|_C)$.

This implies, in particular, that $\mathcal{O}_X(D)$ is a nef and big line bundle on $X$.

– Since $X$ has a nef and big line bundle which is base point free near $C_1$, we may suppose that $X$ is projective.

Let $m \geq 2$. Then we have an exact sequence

$$0 \to H^0(X, \mathcal{O}((1-m)D)) \to H^0(X, \mathcal{O}(D)) \to H^0(D_m, \mathcal{O}(D)) \to H^1(X, \mathcal{O}((1-m)D)).$$

– Since $\mathcal{O}(D)$ is nef and big, by Theorem 5.15 we have that $H^0(X, \mathcal{O}((1-m)D)) = H^1(X, \mathcal{O}((1-m)D)) = \{0\}$. Consequently we have that, for every $m \geq 2$ the natural restriction map

$$H^0(X, \mathcal{O}(D)) \to H^0(D_m, \mathcal{O}(D))$$

is an isomorphism.

– Because of the isomorphisms above, by the Theorem of formal functions 5.6, the restriction map

$$H^0(X, \mathcal{O}(D)) \to H^0(\mathcal{X}, \mathcal{O}(D))$$

is an isomorphism.

– The morphism $\mathcal{X} \to \hat{\mathbb{P}}_C^2 \to \mathbb{P}^2$ is given, by functoriality, by a subspace of dimension three $V$ of $H^0(\mathcal{X}, \hat{g}^*(\mathcal{O}(1)))$ and a surjective map

$$V \otimes \mathcal{O}_\mathcal{X} \to \hat{g}^*(\mathcal{O}(1))$$

– The linear system $Sym^d(V) \hookrightarrow H^0(\mathcal{X}, \mathcal{O}(D))$ defines a morphism

$$f : \mathcal{X} \to \mathbb{P}(Sym^d(V))$$

which factorises through the Veronese embedding $v_d : \mathbb{P}^2 \to \mathbb{P}(Sym^d(V))$.

– The linear system space $Sym^d(V)$ defines also, via the isomorphism 5.40 a linear system on $X$ which is without fixed points around $D$. 
The rational map $\varphi : X \dashrightarrow \mathbb{P}(\text{Sym}(V))$ associated to the linear system $\text{Sym}^d(V)$ is defined in a neighbourhood of $D$. Consequently, solving the indeterminacies of it, we may suppose that it is defined everywhere.

Consequently we have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{X} & \xrightarrow{\varphi} & X \\
\downarrow & & \downarrow \\
\mathbb{P}(\text{Sym}(V)) & & 
\end{array}
\]

Since the morphism 5.42 factorisez through $\mathbb{P}^2$, we eventually get that the morphism $\varphi$ defines a morphism $g : X \to \mathbb{P}^2$ which extends $f$. In particular $g$ is an algebraic morphism to $\mathbb{P}^2$ which is étale near $C$.

The branch locus of $g$ must be a divisor which should cut $C$ somewhere. But, since it is étale around $C$, we eventually find a contradiction.

Consequently such a $X$ cannot exist.

References


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